EQUATIONS OF MOTION OF A FREE BELL AND CLAPPER

Salvador Ivorra*(1), María José Palomo (2), and Gumersindo Verdú (2)
(1) Departamento de Mecánica de los Medios Continuos y Teoría de Estructuras
(2) Departamento de Ingeniería Química y Nuclear
Universidad Politécnica de Valencia
C.P. 46022 – Valencia, Spain

Introduction

In the design, project or restoration phases of a belfry it is necessary to know the time variable forces introduced by the movement of the bells. Depending on their swinging speed and their unbalance they can present possible dynamic interactions with the supports structure. The three more extended systems of making the bells sound are the Central European -fig. 1 -, the English - fig. 2 - and the Spanish -fig. 3 -. In the first of them the bells tilt on their axis, this is a very unbalanced system. In the case of the English system the bells rotate around their axis to a complete circle, each revolution in one direction and the bells are also very unbalanced. In these cases the bells are specially located inside the tower, housed in structures designed to the effect. In the Spanish system the ringing bells rotate in a complete circle and continue in the same sense. They are very balanced and are directly anchored on the tower windows.

Statement of the Problem

The Central European and English systems used to swing bells have been studied by different authors. For the English system the work of Heyman, J. & Therefall (1976) is a classic with respect to the methodology for the determination of the inertia and the center of gravity of a bell. Beside these work, the work Wilson & Selby (1993, 1997) analyzes several towers subjected to the dynamic action originated by the bells. Diverse authors have analyzed the Central European system and its study is even standardized through the norm DIN 4178. Niederwanger (1997) and Schutz (1994) present the last studies referred to this system. There are no references to the Spanish system, and thus the sense of this study.

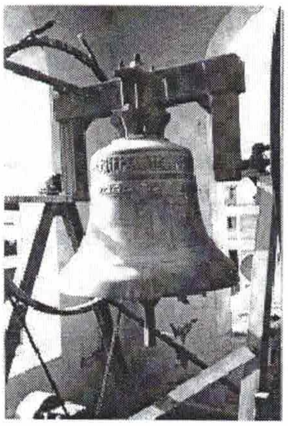

Fig. 1.- Central European system to swing bells

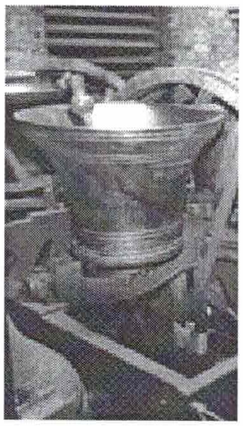

Fig. 2.- English system to swing bells

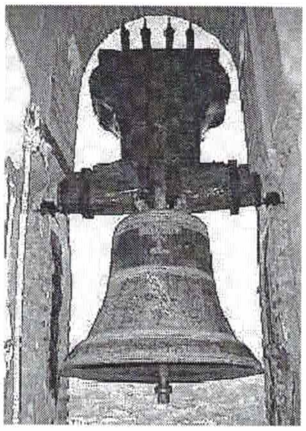

Fig. 3.- Spanish system to swing bells

** corresponding author: Sivorra@mes.upv.es tel: 034 96 387 70 07 ext. 76742; fax: 034 96 387 96 79*

Figure 4 presents the section of a swinging bell, inside it we can observe the clapper pivoting around an axis inside the bell. Supposing the mass of the bell and that of the clapper concentrated respectively on the points G_1 and G_2, we will have a system with two degrees of freedom and therefore two modes of vibration. The similarity between this system and that of the double pendulum is evident, for it the position of the points G_1 and G_2 will come given for:

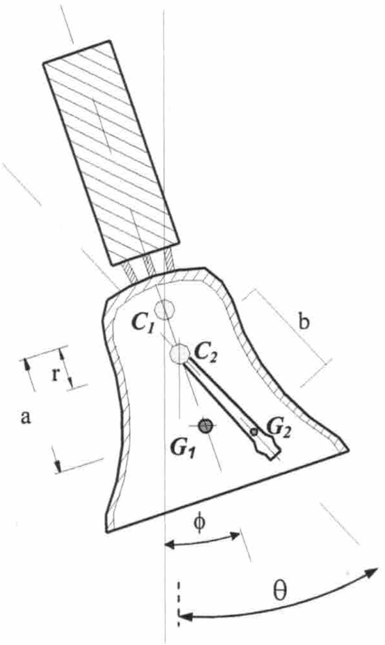

Fig. 4.- Bell simplificate model

$$\left.\begin{aligned} x_1 &= a\cdot \mathrm{sen}\phi \\ y_1 &= -a\cdot\cos\phi \end{aligned}\right\} \qquad \left.\begin{aligned} x_2 &= r\cdot\mathrm{sen}\phi + b\cdot\mathrm{sen}\theta \\ y_2 &= -r\cdot\cos\phi - b\cdot\cos\theta \end{aligned}\right\} \qquad (1)$$

Where:

G_1	Bell centre of gravity
G_2	Clapper centre of gravity
C_1	Axis of rotation of bell
C_2	Axis of rotation of clapper
a	Centre of gravity of the bell
b	Centre of gravity of the clapper from C_2
r	Separation of the two swing axes (usually positive)
θ	Angular offset of clapper from the central axial of bell
ϕ	Angular offset of bell from the downward vertical
g	Acceleration due to gravity
t	Time
M	Mass of bell and yoke
m	Mass of clapper

The potential energy of the system is:

$$U = M\cdot g\cdot y_1 + m\cdot g\cdot y_2 = -(M\cdot g\cdot a + m\cdot g\cdot r)\cdot\cos\phi - m\cdot g\cdot b\cdot\cos\theta \qquad (2)$$

The kinetic energy of the system is:

$$T = \frac{1}{2}\cdot M\cdot v_1^2 + \frac{1}{2}\cdot m\cdot v_2^2 = \frac{1}{2}\cdot M\cdot\left(a\cdot\dot{\phi}\right)^2 + \frac{1}{2}\cdot m\cdot\left((r\cdot\dot{\phi})^2 + 2\cdot r\cdot b\cdot\dot{\phi}\cdot\dot{\theta}\cdot\cos(\theta-\phi)\right) \qquad (3)$$

Applying the Lagrange Equation the movement differential equations of the bell and clapper can be determined:

$$(4)\quad \frac{\partial L}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0; \qquad (5)\quad \frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 \qquad \text{donde} \quad L = T\text{-}U$$

Since the mass of the clapper is approximately 0.25 times the mass of the bell these equations they can be simplified to:

$$I_b\cdot\ddot{\phi}(t) + M\cdot g\cdot a\cdot\sin\phi(t) = 0 \qquad (6)$$

$$I_c\cdot(\ddot{\phi}+\ddot{\theta}) + m\cdot g\cdot b\cdot\sin(\phi+\theta) + m\cdot r\cdot b\cdot\sin(\theta)\cdot\dot{\phi}^2 + m\cdot r\cdot b\cdot\cos(\theta)\cdot\ddot{\phi} = 0 \qquad (7)$$

Where I_b and I_c are the moment of polar inertia of the bell and clapper. The equation (6) represents the differential equation of the movement of the bell; we can observe that it is reduced to the equation of the movement of a compound pendulum. Equation (6) whose solution comes determined by the expression:

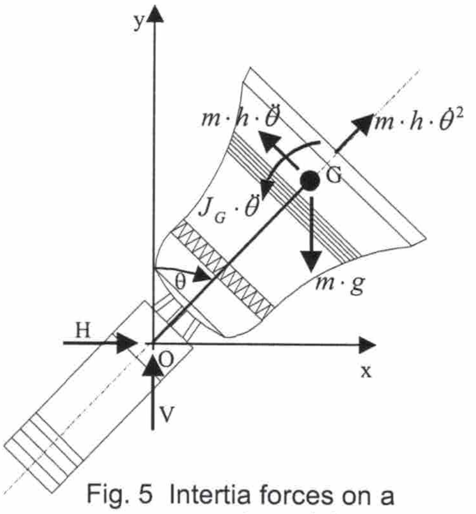

Fig. 5 Intertia forces on a compound pendulum

$$\phi(t) = 2 \cdot \arcsin\left[\beta \cdot \mathrm{sn}\left(\sqrt{\frac{g \cdot M \cdot a}{I_b}} \cdot t\right)\right] \quad (8)$$

sn = Elliptic Jacobian function
β = Constant depending on the initial conditions

$$\beta = \sqrt{\frac{\left(\omega_0^2 + \frac{2 \cdot g \cdot M \cdot a}{I_b} \cdot (1 - \cos\omega_0)\right) \cdot I_b}{4 \cdot g \cdot M \cdot a}} \quad (9)$$

Where the initial speed of free bell is the initial speed indicated in the Table 1, for each bell:

$$\omega_0 = n \cdot \pi / t \ \text{(rad/s)} \quad (10)$$

n = number of clapper blows

t = time (in s).

As consequence of this oscillatory movement, each bell introduces on its supports a horizontal and a vertical forces variable with the time. The equations (11) and (12) represents approximately the horizontal and vertical forces:

$$H(t) = M \cdot a \cdot \left[(\dot{\phi})^2 \cdot \sin\phi(t) - \ddot{\phi} \cdot \cos\phi(t)\right] \quad (11)$$

$$V(t) = -M \cdot g - M \cdot a \cdot \left[(\dot{\phi})^2 \cdot \cos\phi(t) + \ddot{\phi} \cdot \sin\phi(t)\right] \quad (12)$$

Numerical Examples

We have selected diverse bells of the Central European system, -DIN 4178 - English system -Lund (1994) - and Spanish system -laboratory tests -. Table 1,2 and 3 presents their characteristics and a comparison between the variables horizontal and vertical forces introduced by each one of them on their supports. These values may be increased in function of the dynamic characteristics of the tower in which they are located. To be able to compare the models, the values of the presented horizontal and vertical forces are divided for the total weight of the bell.

Table 1: Spanish system

	Unbalance (m)	Weight (N)	Swing velocity (rad/s)	Adim. Horizontal force.	Adim.Vertical force
(1s)	0.14	2000	3.97	0.38	1.46
(2s)	0.02	4300	3.14	0.035	1.042
(3s)	0.08	6270	2.82	0.12	1.14
(4s)	0.08	8260	2.61	0.09	1.1
(5s)	0.03	11020	2.2	0.033	1.03

Table 2: English system

	Unbalance (m)	Weight (N)	Swing velocity (rad/s)	Adim. Horizontal force.	Adim.Vertical force
(1e)	0.33	2096	1.26	3.33	6.02
(2e)	0.49	4318	1.304	4.48	7.77
(3e)	0.307	6505	1.29	1.5	3.28
(4e)	0.27	8215	1.30	1.19	2.79
(5e)	0.424	9662	0.96	2.10	4.16

Table 3: Central European system

	Unbalance (m)	Weight (N)	Swing velocity (rad/s)	Oscillation angle	Adim. Horizontal force	Adim.Vertical force
(1c)	0.36	2100	3.4	71°	2.23	5.77
(2c)	0.46	4300	3.14	67°	1.99	5.33
(3c)	0.52	6000	2.98	65°	1.87	5.13
(4c)	0.58	8000	2.88	63°	1.76	4.94
(5c)	0.6	10000	2.77	62°	1.69	4.83

We can see in Tables 1, 2 and 3 the differences between one system and another: bells of similar weight have unbalances very different. In a same way the oscillation speeds are also very unequal. These results take to very different values in the horizontal and vertical forces that these bells introduce on their supports.

Conclusions

The values presented in tables 1, 2 and 3 allow affirming that the Spanish system to swing bells transmits smaller forces on the support structure than other does. In this system bells oscillate to complete circle at similar speeds to those of the Central European system, but in the first system bells have a degree of balance very superior to the second. Don't have sense try to substitute a system for another in the different areas where they are used because this would affect to the sonority of the bells, where moreover the mechanical problems, social and anthropological factors also coexist and they are not reason of this study.

References

Heyman, J. & Therefall, B.D. (1976) Inertia Forces due to Bell Ringing, International Journal of Mechanical Sciences, 18, 161-164.

DIN 4178: “Glockentürme: Berechnung und Ausfhrung”; 1978

Niederwanger, G (1997). Structural Repair of Damaged Old Bell Towers Based on Dynamic Measurements. Structural studies, Repairs and Maintenance of Historical Buildings. Ed. Sanchez. Computational Mechanical Publications. Southampton, 321-330

Schutz, K. G.. (1994) Dynamische Beanspruchung von Glockentürmen. Bauingenieur 69, Springer-Verlag , 211-217

Wilson, J.M., Selby A.,. (1993) *Engineering a Cathedral*, London, Thomas Telford Ltd., 77-100

Selby, A. R. and Wilson J. M. (1997), Dynamic behaviour of masonry church bell towers, Worldwide Advances in Structural Concrete and Masonry, Proceedings of the CCMS Symposium, Chicago, April 15-18, (Edited by A. E Schultz and S. L. McCabe), ASCE, New York, 188-199.